

GROUPOIDS ASSOCIATED WITH ENDOMORPHISMS

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ABSTRACT. To a compact Hausdorff space which covers itself, we associate an r -discrete locally compact Hausdorff groupoid. Its C^* -algebra carries an action of the circle allowing it to be regarded as a crossed product by an endomorphism and as a generalization of the Cuntz algebra O_p . We consider examples related to coverings of the circle and of a Heisenberg 3-manifold.

Given a compact Hausdorff space X and a continuous surjective map $\sigma: X \rightarrow X$, we want to associate to the pair (X, σ) a C^* -algebra $C^*(X, \sigma)$ which, in the case σ is a homeomorphism, coincides with the well-known crossed product $C(X) \rtimes_{\sigma} \mathbb{Z}$.

In this paper we consider the case of a covering $\sigma: X \rightarrow X$ and $C^*(X, \sigma)$ will be defined via a groupoid that generalizes the construction of the Cuntz groupoid O_p considered by Renault in [Re]. In that case X is a unilateral sequence space and σ is the unilateral shift. We start with some definitions and notation that make sense in greater generality.

Definition. Given a set X and a map $\sigma: X \rightarrow X$, let

$$\Gamma = \{(x, n, y) \in X \times \mathbb{Z} \times X \mid \exists k, l \geq 0, n = l - k, \sigma^k x = \sigma^l y\}$$

with the set of composable pairs

$$\Gamma^{(2)} = \{((x, n, y), (w, m, z)) \in \Gamma \times \Gamma \mid w = y\}.$$

The multiplication and inversion are given by

$$(x, n, y)(y, m, z) = (x, m + n, z) \quad \text{and} \quad (x, n, y)^{-1} = (y, -n, x).$$

The range and source maps are

$$r(x, n, y) = (x, 0, x), \quad s(x, n, y) = (y, 0, y).$$

With these operations Γ becomes a groupoid with the unit space identified with X . The equivalence relation associated to Γ will be denoted by R , where

$$R = \{(x, y) \in X \times X \mid \exists k, l \geq 0, \sigma^k x = \sigma^l y\},$$

and the isotropy group bundle is

$$I = \{(x, n, x) \in X \times \mathbb{Z} \times X \mid \exists k, l \geq 0, n = l - k, \sigma^k x = \sigma^l x\}.$$

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For $k \geq 0$ we consider

$$R_k := \{(x, y) \in X \times X \mid \sigma^k x = \sigma^k y\}$$

and

$$R_\infty := \bigcup_{k \geq 0} R_k.$$

Note that $R_\infty \subset R$.

Theorem 1. *Suppose $\sigma: X \rightarrow X$ is a covering. Then Γ , defined above, carries a topology making it an r -discrete locally compact Hausdorff groupoid with the Haar system given by the counting measures.*

Proof. Consider on R_k the induced topology from $X \times X$. Then R_k is a principal r -discrete groupoid since σ^k is a local homeomorphism (see [Ku1]). Consider on $R_\infty = \bigcup_{k \geq 0} R_k$ the inductive limit topology. Since the R_k are compact as topological spaces, R_∞ is a hyperfinite relation in the terminology of Kumjian (see [Ku2]). Hence R_∞ is a principal r -discrete groupoid and X , identified with the diagonal in $X \times X$, is open in R_∞ . Let

$$\tilde{X} = \left\{ (x_i) \in \prod_{i \geq 0} X_i \mid X_i = X, \sigma(x_{i+1}) = x_i \forall i \geq 0 \right\}$$

be the projective limit space and let $\pi_0: \tilde{X} \rightarrow X$ the projection onto the first component. With the product topology, $\tilde{X}$ becomes a compact Hausdorff space, and π_0 a continuous open map. The map σ induces a homeomorphism $\tilde{\sigma}: \tilde{X} \rightarrow \tilde{X}$,

$$\tilde{\sigma}(x_0, x_1, x_2, \dots) = (\sigma x_0, x_0, x_1, \dots),$$

with inverse

$$\tilde{\sigma}^{-1}(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots),$$

such that $\pi_0 \circ \tilde{\sigma} = \sigma \circ \pi_0$ (see [Br]). Let $\tilde{R}_\infty = (\pi_0 \times \pi_0)^{-1}(R_\infty)$ equipped with the preimage of the inductive limit topology. Then

$$\tilde{R}_\infty = \{((x_i), (y_i)) \in \tilde{X} \times \tilde{X} \mid \exists k \geq 0, \sigma^k x_0 = \sigma^k y_0\}$$

is a principal r -discrete groupoid with unit space identified with $\tilde{X}$. The map $\tilde{\sigma}$ induces an automorphism of $\tilde{R}_\infty$ by the formula

$$\tilde{\sigma}((x_i), (y_i)) = (\tilde{\sigma}(x_i), \tilde{\sigma}(y_i)),$$

and one can consider the semidirect product $\tilde{\Gamma} = \tilde{R}_\infty \rtimes_{\tilde{\sigma}} \mathbb{Z}$ (see [Re]). Two elements of $\tilde{\Gamma}$, written as $((x_i), n, (y_i))$ and $((w_i), m, (v_i))$ with $((x_i), (y_i)), ((w_i), (v_i)) \in \tilde{R}_\infty$, are composable iff $(w_i) = \tilde{\sigma}^{-n}(y_i)$, $(v_i) = \tilde{\sigma}^{-n}(z_i)$ and then the product is given by the formula

$$((x_i), n, (y_i)) \cdot (\tilde{\sigma}^{-n}(y_i), m, \tilde{\sigma}^{-n}(z_i)) = ((x_i), n + m, (z_i)).$$

The inverse of $((x_i), n, (y_i))$ is $(\tilde{\sigma}^{-n}(y_i), -n, \tilde{\sigma}^{-n}(x_i))$. With the induced topology from $\tilde{R}_\infty \times \mathbb{Z}$, $\tilde{\Gamma}$ becomes an r -discrete locally compact groupoid. Note that $\Gamma = \pi(\tilde{\Gamma})$, where

$$\pi: \tilde{X} \times \mathbb{Z} \times \tilde{X} \rightarrow X \times \mathbb{Z} \times X$$

is given by the formula

$$\pi((x_i), y, (y_i)) = (x_0, n, \pi_0(\tilde{\sigma}^{-n}(y_i))),$$

and that π is a groupoid homomorphism. With the quotient topology, Γ becomes an r -discrete locally compact groupoid. It turns out that a basis of open sets for the topology of Γ consists of the subsets of the form $U \times \{n\} \times V$, where $n \in \mathbb{Z}$, U, V are open in X , $\sigma^n|_V: V \rightarrow \tau(U)$ (respectively $\sigma^{-n}|_U: U \rightarrow \rho(V)$) is a homeomorphism for $n \geq 0$ (respectively $n \leq 0$), and τ, ρ are suitable deck transformations for suitable powers of σ . The equivalence relation R with the trace of this topology was considered also by Vershik and Arzumian (see [ArVe]). Note that, by construction, Γ has a (left) Haar system given by the counting measures. $\square$

Definition. If X is compact and $\sigma: X \rightarrow X$ is a covering map, we define $C^*(X, \sigma)$ to be $C^*(\Gamma)$.

Observe that, in the case σ is 1-1, Γ is the transformation group groupoid $X \times \mathbb{Z}$ and $C^*(\Gamma) \simeq C(X) \rtimes_{\sigma} \mathbb{Z}$. But note that when σ is not 1-1 (the case in which we are interested), $C^*(\Gamma)$ is more closely allied to a crossed product of $\mathbb{Z}_+$ on the noncommutative algebra $C^*(R_{\infty})$. To see this and in order to study the structure of $C^*(X, \sigma)$, let $c: \Gamma \rightarrow \mathbb{Z}$,

$$c(x, n, y) = -n.$$

Then c is a 1-cocycle, which induces an S^1 -action on $C^*(\Gamma)$ via the formula

$$(\lambda f)(x, n, y) = \lambda^{-n} f(x, n, y), \quad f \in C_c(\Gamma), \lambda \in S^1,$$

where $C_c(\Gamma)$ are the compactly supported functions on Γ . The fixed point algebra is $C^*(R_{\infty})$, which appears as an inductive limit of $C^*(R_k)$, $k \geq 0$. Kumjian observed in [Ku1] that $C^*(R_k) \simeq C(X) \rtimes G_k$, where G_k is the group of deck transformations of the covering $\sigma^k: X \rightarrow X$.

Definition. The orbit of $x \in X$ is defined to be

$$O(x) = \bigcup_{k \geq 0} \sigma^{-k} \sigma^k x,$$

where

$$\sigma^{-k} y = \{z \in X \mid \sigma^k z = y\}.$$

We say that σ *minimal* if each orbit is dense and that σ is *essentially free* if

$$\{x \in X \mid \forall k, l \geq 0, \sigma^k x = \sigma^l x \Rightarrow k = l\}$$

is dense in X .

Proposition. If σ is minimal, then $C^*(R_{\infty})$ is simple. Moreover, if σ is essentially free, then $C^*(\Gamma)$ is also simple.

Proof. Since σ is minimal and

$$O(x) = \{y \in X \mid (x, y) \in R_{\infty}\} \subset \{y \in X \mid (x, y) \in R\},$$

it follows that there are no nontrivial open invariant subsets. If σ is also essentially free, then the groupoid Γ is essentially principal in the sense of Renault (see Definition II.4.3 of [Re]). Now apply Proposition II.4.6 of [Re], where the ideals of an essentially principal groupoid are characterized. $\square$

The covering map σ induces a $*$ -endomorphism of $C^*(R_\infty)$, denoted also by σ , via the formula

$$(\sigma f)(x, y) = \frac{1}{p} f(\sigma x, \sigma y), \quad f \in C_c(R_\infty),$$

where p is the index of the covering.

Indeed the fact that this gives an endomorphism follows from these calculations:

$$(\sigma(fg))(x, y) = \frac{1}{p} (fg)(\sigma x, \sigma y) = \frac{1}{p} \sum_{z \sim \sigma x} f(\sigma x, z) g(z, \sigma y)$$

and

$$\begin{aligned} ((\sigma f)(\sigma g))(x, y) &= \sum_{x \sim w} (\sigma f)(x, w) (\sigma g)(w, y) \\ &= \frac{1}{p^2} \sum_{x \sim w} f(\sigma x, \sigma w) g(\sigma w, \sigma y) = \frac{1}{p} \sum_{\sigma x \sim z} f(\sigma x, z) g(z, \sigma y), \end{aligned}$$

where $z = \sigma w$ and $x \sim y$ means $(x, y) \in R_\infty$.

One can check easily that the induced endomorphism is never unital (unless σ is a homeomorphism). However, more important for our purposes, σ is induced by a nonunitary isometry and this allows us to represent $C^*(X, \sigma)$ as the crossed product of $C^*(R_\infty)$ by this endomorphism in the sense of Paschke (see [Pa1, Pa2]). In his notation, $B = C^*(\Gamma)$, $A = C^*(R_\infty)$ and the spectral subspace E_1 is generated by functions supported on $\{(x, n, y) \in \Gamma \mid n = -1\}$. Let

$$v(x, n, y) = \begin{cases} 1/\sqrt{p} & \text{if } n = -1 \text{ and } y = \sigma x, \\ 0 & \text{otherwise.} \end{cases}$$

Then $v \in E_1$,

$$(v^*v)(x, n, y) = \begin{cases} 1 & \text{if } x = y \text{ and } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus v is an isometry in B and

$$(vv^*)(x, n, y) = \begin{cases} 1/p & \text{if } \sigma x = \sigma y \text{ and } n = 0, \\ 0 & \text{otherwise} \end{cases}$$

is a projection in A , different from 1. Moreover, $\sigma f = v f v^*$. These calculations lead to

Theorem 2. Suppose $\sigma: X \rightarrow X$ is minimal. Then we have an exact sequence

$$\begin{aligned} \cdots \rightarrow K_0(C^*(R_\infty)) &\xrightarrow{\sigma_* - \text{id}} K_0(C^*(R_\infty)) \xrightarrow{i_*} K_0(C^*(X, \sigma)) \rightarrow \\ &\rightarrow K_1(C^*(R_\infty)) \xrightarrow{\sigma_* - \text{id}} K_1(C^*(R_\infty)) \xrightarrow{i_*} K_1(C^*(X, \sigma)) \rightarrow \cdots \end{aligned}$$

where $i: C^*(R_\infty) \hookrightarrow C^*(X, \sigma)$ is the inclusion map and id is the identity on K -theory.

Proof. Using the fact that $\sigma f = v f v^*$, it follows that $\sigma(C^*(R_\infty))$ is a corner in $C^*(R_\infty)$. Since $C^*(R_\infty)$ is simple, it is a full corner and we may apply Theorem 4.1 of [Pa2]. Notice that

$$\sigma: C^*(R_\infty) \rightarrow \sigma(C^*(R_\infty))$$

is an isomorphism, therefore σ_* is an isomorphism at the level of K -theory. $\square$

We turn now to some examples that illustrate the applicability and the limitations of our analysis.

Example 1. Let $X = \prod_{i \geq 0} X_i$, where $X_i = \{1, 2, \dots, p\}$ for each $i \geq 0$, with product topology, and let $\sigma: X \rightarrow X$ be the unilateral shift,

$$\sigma(x_0, x_1, \dots) = (x_1, x_2, \dots).$$

Then $\sigma: X \rightarrow X$ is a p -fold covering. In this case

$$C^*(R_k) \simeq M_{p^k}(C(X)),$$

R_∞ is the Glimm equivalence relation,

$$C^*(R_\infty) \simeq UHF(p^\infty),$$

and $C^*(X, \sigma)$ is the Cuntz algebra O_p . Note that σ is essentially free since the points in X with trivial isotropy are the nonperiodic sequences, which are dense in X . The fact that $UHF(p^\infty)$ and O_p are simple is well-known.

Example 2. Let $A = (A(i, j))$ be a $p \times p$ matrix of 0's and 1's, let X be as in example 1 and let

$$X_A = \{(x_i) \in X \mid A(x_i, x_{i+1}) = 1\}.$$

Define $\sigma_A: X_A \rightarrow X_A$ to be the restriction of σ to X_A . For simplicity, we require that

$$\sum_{i=1}^p A(i, j) = q \quad \forall j$$

for some $q \geq 2$, so that σ_A is a q -fold covering. (Without this condition, we would consider different q 's for different pieces of X_A). Then

$$C^*(R_k) \simeq M_{q^k}(C(X_A)),$$

$C^*(R_\infty)$ is an AF -algebra, and $C^*(X_A, \sigma_A)$ is the Cuntz-Krieger algebra O_A . If the matrix A is aperiodic, then σ_A is minimal and essentially free, so that $C^*(R_\infty)$ and O_A are simple (again a well-known result).

Example 3. Let $X = \mathbb{T}$, $\sigma: \mathbb{T} \rightarrow \mathbb{T}$, $\sigma(x) = x^p$, $p \geq 2$. In this case $C^*(R_k) \simeq M_{p^k}(C(\mathbb{T}))$ (see [Ku1]), and $C^*(R_\infty)$ is the Bunce-Deddens algebra of type p^∞ . It is known that

$$K_0(C^*(R_\infty)) \simeq \mathbb{Z}[1/p], \quad K_1(C^*(R_\infty)) \simeq \mathbb{Z}$$

(see [B1]) and it is easy to see that σ is minimal. Thus one may apply the exact sequence in Theorem 2 to compute the K -theory of $C^*(\mathbb{T}, \sigma)$. In order to determine σ_* , observe that $\sigma(C^*(R_k)) \subset C^*(R_{k+1})$ and it will be enough to know the maps at the level of K -theory induced by

$$C^*(R_0) = C(\mathbb{T}) \xrightarrow{\sigma} M_p(C(\mathbb{T})) = C^*(R_1).$$

Borrowing notation from [Ku1], $l^2(\sigma)$ is a $C(\mathbb{T})$ -Hilbert module via the inner product

$$(f|h)(x) = \sum_{\sigma y=x} \bar{f}(y)h(y), \quad f, h \in C(\mathbb{T}),$$

with basis $g_j(x) = x^j/\sqrt{p}$, $j = 0, 1, \dots, p-1$. Furthermore, $C^*(R_1) \simeq \mathcal{K}(l^2(\sigma))$. Since $f \in C(R_1)$ may be viewed as an element of $\mathcal{K}(l^2(\sigma))$ via the formula

$$(fg)(x) = \sum_y f(x, y)g(y), \quad g \in l^2(\sigma),$$

one can identify f with the matrix (f_{lj}) , where $f_{lj} = (g_l | fg_j)$. So, letting $x = e(t) = \exp(2\pi it)$, we see that

$$f_{lj}(e(t)) = \frac{1}{p} \sum_{k,m=0}^{p-1} e((t(j-l) + k(j-l) + mj)/p) f(e((t+k)/p)),$$

$$e((t+m+k)/p)).$$

Following these identifications, the projection $\sigma(1) = vv^*$ corresponds to $e_{11} \otimes 1$ in $M_p(C(\mathbb{T}))$, so that

$$Z = K_0(C(\mathbb{T})) \xrightarrow{\sigma_*} K_0(M_p(C(\mathbb{T}))) = Z$$

is the identity map. If we denote by u the unitary

$$u(x, n, y) = \begin{cases} x & \text{if } y = x \text{ and } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

which generates $C(\mathbb{T})$, then vvv^* corresponds in $M_p(C(\mathbb{T}))$ to $e_{11} \otimes u$ so that

$$Z = K_1(C(\mathbb{T})) \xrightarrow{\sigma_*} K_1(M_p(C(\mathbb{T}))) = Z$$

is again the identity map. Therefore $\sigma_*: K_0(C^*(R_\infty)) \rightarrow K_0(C^*(R_\infty))$ is the multiplication by $1/p$, and $\sigma_*: K_1(C^*(R_\infty)) \rightarrow K_1(C^*(R_\infty))$ is the identity map. We get the exact sequences

$$0 \rightarrow Z_{p-1} \rightarrow K_0(C^*(\mathbb{T}, \sigma)) \rightarrow Z \rightarrow 0$$

and

$$0 \rightarrow Z \rightarrow K_1(C^*(\mathbb{T}, \sigma)) \rightarrow 0,$$

from which it follows that

$$K_0(C^*(\mathbb{T}, \sigma)) \simeq Z \oplus Z_{p-1} \quad \text{and} \quad K_1(C^*(\mathbb{T}, \sigma)) \simeq Z.$$

The isotropy groups

$$I_x = \{n \in \mathbb{Z} | x^{p^k} = x^{p^l}, l - k = n\}$$

are trivial for a dense set of x in $\mathbb{T}$, therefore σ is essentially free and $C^*(\mathbb{T}, \sigma)$ is a simple C^* -algebra. Since the corresponding isometry v is nonunitary, it follows that $C^*(\mathbb{T}, \sigma)$ is infinite.

One can consider coverings of $\mathbb{T}^q$ for $q \geq 2$ to get other simple infinite C^* -algebras.

Example 4. Consider the real Heisenberg group of dimension 3, N . We write the elements of N as triples $(x, y, z) \in \mathbb{R}^3$ with multiplication

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy').$$

Consider also the discrete Heisenberg group

$$L = \{(x, y, z) \in N \mid x, y, z \in \mathbb{Z}\},$$

and let $X = N/L$. This is a compact 3-manifold (a circle bundle over T^2). The elements of X will be written as $[x, y, z]$, $x, y, z \in [0, 1)$. Let

$$s: N \rightarrow N, \quad s(x, y, z) = (px, qy, pqz),$$

where $p, q \geq 2$ are integers. Since $s(L) \subset L$, we get a map

$$\sigma: X \rightarrow X, \quad \sigma[x, y, z] = [px, qy, pqz],$$

which is a p^2q^2 -fold covering (see [Sh]). Invoking the methods of Kumjian [Ku1] again, we observe that a basis for the $C(X)$ -Hilbert module $l^2(\sigma)$ is given by

$$g_{l,m,n}[x, y, z] = \frac{1}{pq} e(lx + my + nz), \quad \begin{aligned} l &= 0, \dots, p-1, \\ m &= 0, \dots, q-1, \\ n &= 0, \dots, pq-1, \end{aligned}$$

where $e(a) = \exp(2\pi ia)$. This allows us to identify $C^*(R_1)$ with $M_{p^2q^2}(C(X))$. Therefore $C^*(R_\infty)$ will be an inductive limit of matrix algebras over $C(X)$.

Observe that σ is minimal and essentially free, so that $C^*(R_\infty)$ and $C^*(X, \sigma)$ are simple. It is known that the computation of $K_*(C(X))$ may be reduced to the computation of $K_*(C^*(L))$, which is done by Anderson and Paschke (see [AnPa]). The computation of the K -theory of $C^*(X, \sigma)$ using Theorem 2 can proceed, at least in principle. However at the moment, the details appear quite complicated.

Remark. If $\sigma: X \rightarrow X$ is a branched covering like $z \rightarrow z^p$ on the closed unit disc D^2 , one can define a locally compact topology on the groupoid Γ using the procedure described above. However, it will no longer be r -discrete. The reason is that the family of measures

$$\lambda^{(0,0)}(f) = 2f(0, 0), \quad \lambda^{(x,x)}(f) = f(x, x) + f(x, -x), \quad x \neq 0,$$

is a Haar system for

$$R_1 = \{(x, y) \in D^2 \times D^2 \mid x^2 = y^2\}$$

which does not consist of counting measures (see [Re]).

Notice that $D^2 \setminus \{0\}$ is an open invariant subset of the unit space D^2 . Groupoids associated with branched coverings will be discussed elsewhere.

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